JOURNAL OF COMBINATORIAL THEORY (A) 14, 167–172 (1973)

Tiling Space with the Aid of the Holomorph

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Communicated by Alan J. Hoffman

Received January, 1971

The notion of a factorization of a group is generalized and a method is presented for obtaining new factorizations from old ones. The results are applied to obtain new fillings of the lattice spaces Z, $Z \oplus Z$ and the cube.

The algebraic approach to the tiling of Euclidean space has been concerned almost exclusively with tiling by translates of a subset of the space. This type of problem generalizes to the algebraic problem of factoring a group. If A and B are subsets of a group G and if every element of the group G is uniquely expressible in the form ab, with $a \in A$ and $b \in B$, then one says that G has the factorization (A, B). In such a case G is the disjoint union of right translates of A and also of left translates of B. In other words, both A and B tile G by translates. This notion appears in the works of Hajos [1], Stein [3], and Rothaus and Thompson [2], the latter paper being the only one concerned with non-Abelian groups. The present paper generalizes some results in [3] to tilings that permit both translations, rotations, and reflections. The results are applied in Section 3 to obtain new tilings of the lattice spaces Z, $Z \oplus Z$ and the cube.

1. DEFINITIONS

The following describes the general notion of a tiling.

DEFINITION. Let S be a set, $F \subseteq S^S$, and $B \subseteq S$. If each element of S is uniquely expressible in the form $f(b)$, $f \in F$, $b \in B$, then B *tiles* S by F, denoted $S = (F, B)$.

Note that the restriction of each $f \in F$ to B is one-to-one. Also observe that the tilings described in the introduction are special cases of the above

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definition, where each element a is replaced by the left translation in the group G induced by a. In the cases that will concern us F will be a left quasi-group. That is, F will be closed under composition and for each pair $f_1, f_2 \in F$, there is a unique $f \in F$ such that $f_1 = ff_2$. In the applications in Section 3, S will be a group and F will be a subgroup of the holomorph of S. Recall that the holomorph, hol(G), of a group G consists of all ordered pairs (g, a) where g runs through G and a runs through the automorphisms of G. The holomorph is a group under the composition defined by $(g_1, a_1)(g_2, a_2) = (g_1 a_1(g_2), g_1 g_2)$. We shall make hol($G$) operate on G by defining $(g, a)\, g_0$ to be $ga(g_0)$, for any $g_0 \in G$ and $(g, a) \in \text{hol}(G)$. Hol(G) is isomorphic to the group of operations on G generated by automorphisms and left translations.

2. Derived Tilings

Theorem 1 describes how one tiling may be derived from another. It will be applied in Section 3.

THEOREM 1. *Let F be a left quasi-group of functions on set S; let $B = \{b_i : i \in I\} \subseteq S$; let $S = (F, B)$; and let $B^* = \{f_i(b_i): i \in I\}$ where $f_i \in F$ for each $i \in I$. Then $S = (F, B^*)$.*

Proof. Suppose $S = (F, B)$ and $f_i \in F$ for each $i \in I$. Let $s \in S$; then $s = f(b_i), f \in F, b_i \in B$ uniquely. Since F is a left quasi-group, there exists for each f_i a unique f' in F such that $f'f_i = f$. Thus $s = f(b_i) = (f'f_i)(b_i) = f'(f_i(b_i))$ where $f \in F$, $f_i(b_i) \in B^*$. Moreover, s has only one such expression of the form $s = f'(f_i(b_i))$, $f' \in F$, $i \in I$. To show this, assume also that $s = f''(f_j(b_j))$, $f'' \in F$, $f_j(b_j) \in B^*$. We shall show $i = j$ and $f' = f''$. Since $(f'f_i)(b_i) = (f''f_j)(b_j)$, it follows by the closure of F and the relation $S = (F, B)$, that $f'f_i = f''f_j$ and $b_i = b_j$. Thus $i = j$, hence $f_i = f_j$. Unique left solvability in F then gives $f' = f''$.

The following theorem shows that the tilings of the preceding theorem exhaust all the tilings in the sense that if $S = (F, B) = (F, C)$, then $C = B^*$ for some choice of the f_i's in F.

THEOREM 2. *If $S = (F, B) = (F, C)$ where F is a left quasi-group of functions on S, and B, $C \subseteq S$, then $C = B^* = \{f_i(b_i): i \in I\}$ for some choice of f_i in F.*

Proof. Suppose $(F, B) = (F, C)$ for some subsets B and C of S and left quasi-group F. First we will see that, for each $b \in B$, there is exactly one $f \in F$ such that $f(b)$ is in C and that each element of C is obtained in this way. Suppose that, for some $b \in B$, there are two distinct elements $f_1, f_2 \in F$ such that $f_1(b) = c_1 \in C$ and $f_2(b) = c_2 \in C$. Since F is a left quasi-group, there exists a unique f' such that $f_2 = f'f_1$. So $c_2 = f'f_1(b) = f'(c_1)$. Thus, for any $f \in F$, $f(c_2) = ff'(c_1)$. But by uniqueness of representation in (F, C) and closure in F we get $c_2 = c_1$; that is, $f_1(b) = f_2(b)$. By uniqueness of representation in (F, B) it follows that $f_1 = f_2$. Consequently each element of C has a unique representation in the form $f_i(b_i)$, $f_i \in F$, $b_i \in B$, as claimed.

The next theorem generalizes the "coloring" Theorem 1.5 of [3] and is useful in deriving one tiling from another (see Example 3).

THEOREM 3. *Let $F \subseteq S^S$ be closed under composition. Then the following conditions* (1), (2), *and* (3) *are equivalent:*

(1) *There exists a function g from S onto index set I such that* (a) *if $g(s_1) = g(s_2)$, then there is a unique $f \in F$ such that $f(s_1) = s_2$*, (b) $g(f(s)) = g(s)$ *for all $f \in F$, $s \in S$,*

(2) *F is a left quasi-group and, for any $f_1, f_2 \in F$, $s \in S$, if $f_1(s) = f_2(s)$ then $f_1 = f_2$,*

(3) *F is a left quasi-group and $S = (F, B)$ for $B \subseteq S$ where B is determined below.*

Proof. First, suppose (1) is satisfied. We show (2). Suppose $f_1(s) = f_2(s)$. By (1b) we know $g(s) = gf_1(s)$. By (1a) there exists a unique $f \in F$ such that $f(s) = f_1(s)$ so $f_2 = f_1$. Thus (2) follows from (1).

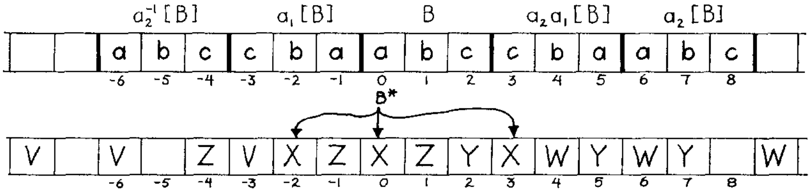
Next, suppose (2). We show (3). Since F is a left quasi-group operating on S, S is partitioned into orbit classes. Choose a set $B \subseteq S$ of representative elements, one from each orbit. Each element of S is in some orbit, hence can be written in the form $f(b)$ for some $f \in F$, $b \in B$. If $f_1(b_1) = f_2(b_2)$ for some $f_1, f_2 \in F$ $b_1, b_2 \in B$, then b_1 and b_2 must be in the same orbit, hence $b_1 = b_2 = b$. By hypothesis $f_1(b) = f_2(b)$ implies $f_1 = f_2$. So $S = (F, B)$.

Last, suppose (3). We show (1). Define $g: s \to B$ by $g(f(b)) = b$. If $g(s_1) = g(s_2) = b$, then $s_1 = f_1(b)$, $s_2 = f_2(b)$ for some $f_1, f_2 \in F$. Since F is a left quasi-group, there exists a unique $f \in F$ such that $ff_2 = f_1$. Thus $s_1 = f_1(b) = ff_2(b) = f(s_2)$ for unique $f \in F$, and (a) is satisfied. Moreover, (b) is also satisfied since for any $s = f(b) \in S$ and for any $f_1 \in F$, $g(f_1(s)) = g(f_1(f(b))) = g(f_1 f(b)) = b = g(s)$.

3. EXAMPLES

The three examples illustrate how the theorems may be applied to get new tilings of the lattice spaces Z, $Z \oplus Z$ and a $4 \times 4 \times 4$ cube.

EXAMPLE 1. Let Z be the group of integers. Let F be the subgroup of hol(Z) generated by $a_1 = (-1, \alpha)$ where $\alpha(n) = -n$ and $a_2 = (6, I)$ where I is the identity. Thus $a_1(n) = -n - 1$ and $a_2(n) = n + 6$. Let $B = \{0, 1, 2\}$. Note that $Z = (F, B)$. Theorem 1 implies that $Z = (F, B^*)$ where $B^* = \{0, a_1(1), a_2a_1(2)\} = \{0, -2, 3\}$. Note that, although B tiles Z by translates, B^* does not (see Figure 1).



$$B^* = \{0, a_1(1), a_2a_1(2)\} = \{0, -2, 3\}$$

FIG. 1. *A* tile B^* derived from tile B.

EXAMPLE 2. Let $Z \oplus Z$ be the plane lattice. Let $B = \{(n, m): 0 \leqslant n \leqslant 2, 0 \leqslant m \leqslant 1\} \subseteq Z^2$. Let a_1 be the identity automorphism on $Z \oplus Z$ and let a_2 be the automorphism defined by $a_2(n, m) = (-n, -m)$. Let F be the subgroup of hol($Z \oplus Z$) generated by $f_1 = ((0,2), a_2)$, $f_2 = ((6,0), a_1)$ and $f_3 = ((3,2), a_1)$. Then $Z \oplus Z = (F, B)$. Figure 2a shows this tiling. The tile B has been labeled with the set I of letters a, b, c, d, e, f in order that the action of F on B can be seen.

By Theorem 1, if we now choose any set B^* consisting of six elements

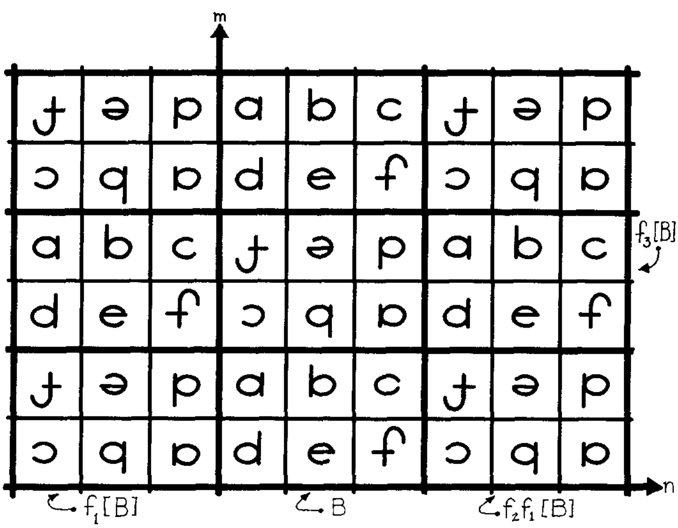


FIG. 2a. A tiling by B via motions F.

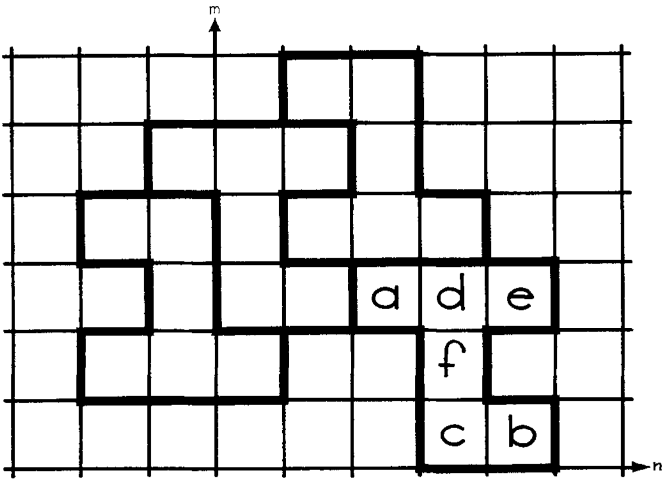


FIG. 2b. *A* tile B^* derived from B.

of $Z \oplus Z$ having the six different labels a, b, c, d, e, f, then $Z \oplus Z = (F, B^*)$. In particular we may choose $B^* = \{f_3(0,0), f_3(1,0), f_3f_1(0,1), f_2f_1(2,0), f_2f_1(2,1), f_2f_1(1,1)\}$. This tiling is shown in Figure 2b.

EXAMPLE 3. Let S consist of 64 unit cubes whose union is a $4 \times 4 \times 4$ cube C as in Fig. 3a. Let a_1 be either of the rotations about the line l_1 by

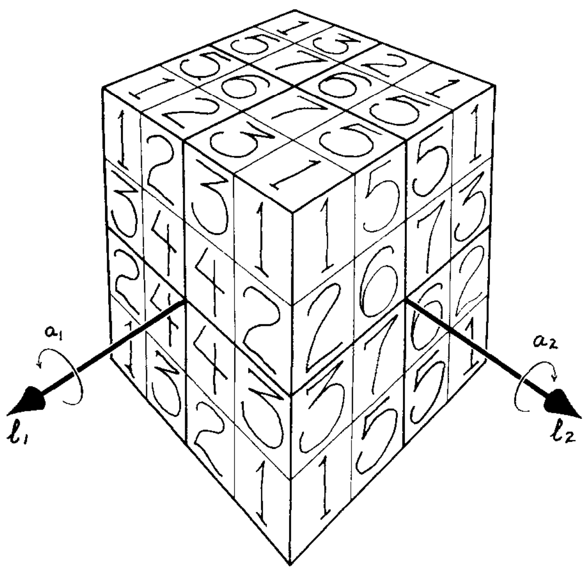


FIG. 3a. Cube C with invariant numbering.

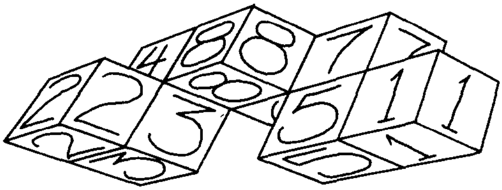


FIG. 3b. *A* derived set B which tiles the cube C.

$\pi/2$ and a_2 be the rotation by π about line l_2 and let F be the group of motions of C generated by a_1 and a_2. Let g assign the integers 1 through 8 to the cubes in S as indicated in Fig. 3a. Note that g satisfies the conditions of Theorem 3. Thus any set B of 8 cubes labeled 1,..., 8 tiles the cube C via motions from F. Figure 3b shows one possibility for such a set B.

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